OPTION PRICING: A TIME SERIES ALTERNATIVE TO BLACK-SCHOLES
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INTRODUCTION:

Your company has just awarded you 100 “stock options”. The exercise price is $120. The current stock price is $110. The options can be exercised anytime in the next six months. Has the company given you something of value? How could you determine what this “value” is?

OVERVIEW:

The purpose of this paper is to introduce a method of valuing stock options using a time series model, and to compare the results of this model with the Black-Scholes formula as well as empirical results from Wall Street transactions. This paper contains three sections:
I) Overview of Stock Options
II) The Black-Scholes Formula
III) Option Pricing using a Time Series

There are several different classes of techniques available to assign a value to options (“option pricing techniques”). The most well known of these is the Black-Scholes formula, which is used by thousands of investors on a daily basis. The purpose of my paper is not to improve on the work of Drs. Black & Scholes (which was deemed worthy of the 1997 Nobel Prize in Economics). Rather, it is to demonstrate a real world application of time series forecasting, and by comparison to Black-Scholes, shed some light on the statistics involved in option pricing. For an overview of forecasting, please refer to Appendix I. As this paper is written for the SAS Institute Conference, the emphasis is naturally on statistics, partly on the decomposition of the Black-Scholes formula, but mostly on the development of a time series model to estimate the value of an option. The time series analysis presented here can also serve as a primer for many other types of forecasting problems (Section Three of this paper details the steps involved in time series forecasting).

SECTION ONE – OVERVIEW OF STOCK OPTIONS:

An “Option” may be defined as the right, but not the obligation, to engage in some pre-defined transaction. The transaction may involve stocks, indices, exchange rates, interest rates, or commodities. An option may be a “call option” or a “put option”. A call option gives the holder the right to “call” an asset (i.e., purchase it) at a pre-determined price (called the “exercise price” or the “strike price”), by a pre-determined date. Conversely, a put option gives the holder the right to “put” an asset (i.e., sell it) at a pre-determined price, by a pre-determined date.

I will focus my attention on call options on non-dividend paying stocks. This is quite a narrow focus; however, the analysis is largely applicable to other types of options as well. For example, much of the analysis for call options applies to put options, and the same holds true for stock options vs. options on other assets. Dividend payments, however, pose quite a challenge for option pricing. As I discuss in more detail in the next section, the assumption of “no dividends” simplifies the calculations enormously. For more details on stock options, please refer to Appendix II.

SECTION TWO – THE BLACK-SCHOLES FORMULA:

Consider a call option on a stock trading at $100 today, with a strike price of $100. Assume that on any given day, the stock price can increase by 5%, or decrease by 5%, each with equal probability. Now assume that the option must be exercised within three days. What are the possible outcomes?

We can calculate the expected value of the stock price as the stock price at each “outcome” multiplied by the probability of each outcome, or 

\[ \frac{1}{8} \times 85.74 + \frac{3}{8} \times 94.76 + \frac{3}{8} \times 104.74 + \frac{1}{8} \times 115.76 = 100 \]

If we expect the stock to
trade exactly at the strike price, it would appear that this call option has no value. Not so. The expected value of a call option is “skewed positive” since it has positive value when the stock price exceeds the strike price, and zero value otherwise. In the preceding example, we can calculate the value of the call option for each possible outcome:

We can calculate the expected value of the call option as \( (1/8 * 0) + (3/8 * 0) + (3/8 * 4.74) + (1/8 * 15.76) = \$3.75 \). One would expect a rational investor on Day 0 to pay anything less than \$3.75 for this option. What would happen if the volatility of the stock price increased? In this example, what if the stock price can increase by 10%, or decrease by 10%, each with equal probability. The upside gets larger while the downside is unchanged (at zero value). The expected value of the stock price in three days is still \$100\), but the expected value of the call option jumps to \$7.48 \) (calculations omitted, follow same logic)! The same volatility that depresses the value of a stock actually enhances the value of a call option on that stock!

This is a highly simplified example. In the real world, the “discrete nature” of this model is violated. A stock price can take more than 2 paths, and it can move in time increments smaller than a day. In the real world, these movements are continuous (and result in a nearly infinite number of possible outcomes). Modeling a nearly infinite number of possible outcomes can be cumbersome. However, in the above example, let’s assume that there are 1,000 incremental movements of the stock price, instead of two. Let’s also assume that each day is made up of 1,000 time increments, instead of just one. Depending on the exact probabilities selected, this new model may have several thousand possible outcomes, instead of just four (as seen in the prior example), and can very closely approximate reality. This “new model” is a standard binomial model, which is quite common in statistics (and the real world). The set of possible outcomes is a discrete binomial distribution that very closely approximates a continuous normal distribution. This model is quite versatile, but also computationally intensive for most reasonable problems. What would be great is a FORMULA for pricing these options that yielded the same result as this binomial model. Such a formula eluded economists and mathematicians for many years, until it was derived by Fischer Black, Myron Scholes, and Robert Merton in 1973. The Black-Scholes formula is valuable primarily because, through some clever mathematics, the results mirror those of a binomial model, but without any approximations and without the massive computation. The Black-Scholes Option Pricing Formula is:

\[
C_0 = S_0 N(d_1) - X e^{-rT} N(d_2)
\]

Where

\[
d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

And where

\( C_0 \) = call option price today
\( S_0 \) = stock price today
\( N(x) \) = probability that a random selection will be less than \( x \). This assumes a normal probability distribution function for the values of \( x \).
\( X \) = exercise price (“strike price”) of the call option.
\( r \) = risk free interest rate
\( T \) = years until the option expires
\( \sigma \) = standard deviation of the annual rate of return on the stock.

What does all of this mean? I won’t attempt to explain the whole formula in detail, but a few points may make this formula more intuitive: First, let’s assume that the option WILL be exercised. What’s the option worth? The holder gets a stock worth \( S_0 \), and has to pay a value of \( X \) in the future. Paying \( X \) in the future is equivalent to paying the present value of \( X \) today. The present value of \( X \) is \( X e^{-rT} \) (assuming continuous compounding). For example, with \( r=10\% \) and \( T=1.0 \) years, \( X e^{-rT} \) is approximately 90% of \( X \) (slightly more than 90% because of the continuous compounding). If exercise is certain, the \( N(x) \) terms (probability of exercise, explained below) are equal to 1. So, if the
option WILL be exercised, it is worth $S_0 - Xe^{rT}$ today. Unfortunately, we don’t know that the option will be exercised. The N(x) factors serve as “probability adjustments”. Notice that the $\ln(S/X)$ term is approximately the percentage amount by which the current price exceeds the strike price. For example, if $S_0=$11 and $X=$10, the current price exceeds the strike price by 10%. The calculation $\ln(S/X)$ is $\ln(11/10)=9.5\%$. This $\ln(S/X)$ term is then scaled up by $[\sigma * T^{1/2}]$ (as $\sigma$ or $T$ increase, the expected value of the option increases). As the volatility or the time to expiration increase, the option is worth more today. The remaining terms of the formula constitute other “scaling factors”, and are not quite so easily explained (at least not by me!).

This formula has some important underlying assumptions:
1) The stock itself pays no dividends
2) $r$ and $\sigma$ are constant
3) the price of the stock is continuous (no “shocks” allowed)

Violation of any of these assumptions will introduce some prediction error to the formula. More recent, and much more elaborate, versions of the model are able to handle dividend payments (such payments can make early exercise attractive, which makes the calculations quite a bit more complex). In practice, assumptions 2 and 3 are always violated, and the user must understand the degree to which these assumptions are still valid. Interest rates and stock price volatility must be projected based on historical data. In the real world, “shocks” are not uncommon, which means that stock prices are not always continuous.

So, what to make of this formula? My initial reaction was: “How can anyone develop a model for option pricing that does not explicitly involve forecasted price movements of the underlying asset?” Nowhere in this complex formula is there any mention of the growth rate of the stock itself. This seems illogical. The reader might be asking: How did these guys wind up with a Nobel Prize? Perhaps the selection committee was faced with a pretty thin pool of candidates that year! It turns out that the expected rate of return is already implied in the current stock price (assuming it is fairly priced). The current stock price, which is known, is a function of risk (measured volatility) and return (implied). So this is a valid model after all!

Now let’s put this model to the test. I have 8 years of daily stock price history for AMR Corporation (parent of American Airlines), through April 24, 1998. This stock was chosen because it is a large stable company that does not pay dividends. I want to know the value of a call option on this stock, on April 24, 1998, with a strike price of $150, an interest rate of 5%, and an expiration date in October. (Note: expiration dates are quoted by month only. The implied expiration date is the Thursday before the 3rd Saturday of that month). I simply plug the appropriate numbers into the formula, and the result is $13.82. According to the Black-Scholes model, on April 24, 1998, a rational investor would have been willing to pay $13.82 for the “option” to buy one share of AMR on (or before) October 15, 1998, for $150. How good is this model? On April 24, 1998, a rational investor WAS willing to pay $16.00 for the “option” to buy one share of AMR on (or before) October 15, 1998, for $150 (based on actual trading that day).

Why the discrepancy? I believe it’s due to the expectations about volatility. The “volatility” I used in the calculation was simply the historical standard deviation of the yearly returns (=25.5%). Based on an option price of $16, it appears that Wall Street expects the volatility of this stock to be above its historical average over the next 6 months. Note: the sigma in the Black-Scholes formula is NOT the historical volatility, but rather the projected volatility. The user should select a time frame of historical data that is most predictive of the future volatility (this clearly calls for some insight).

SECTION THREE – OPTION PRICING USING A TIME SERIES:

This option pricing technique is somewhat more involved than the Black-Scholes model. Again, the purpose here is to gain some understanding of the statistics involved in option pricing. Recall the binomial model discussed in the last section. Using this model for option pricing involved three steps: 1) calculating a series of probabilities to estimate the future movements of the stock price, 2) for each “outcome” of the resulting binomial distribution, assigning an option value of MAX [forecasted stock price –
exercise price, 0], and 3) calculating the price of the option as the sum, over all possible outcomes, of the option value at each outcome multiplied by the probability of that outcome’s occurrence. Step one involves some estimation and lots of computation, while steps two and three are quite straightforward.

The technique I introduce in this paper, Option Pricing Using A Time Series, simply takes these three steps and substitutes a time series forecast for step one. Once we have a forecast for the stock price (a distribution, not simply a point estimate), we can apply steps two and three to calculate the expected value of the option. This expected value should be the current option price – representing what a rational investor is willing to pay today.

So the problem of calculating an option price is now reduced to the problem of generating a forecast for the stock price. How should we develop a forecast? Assuming that the past can help us predict the future (generally, but not always, a valid assumption), we can use some form of a time series model. [Note: if the past has no predictive value, a causal model might still work – see Appendix I]. Now is a good time for me to emphasize that time series forecasting, while largely quantitative in nature, also has qualitative aspects. If this were not true, there would be a multitude of “turn-key” forecasting packages on the market, where a string of data and a click of the “OK” button would produce a forecast. No self-respecting statistician would utilize such an inflexible system.

This example of AMR stock price forecasting provides an excellent demonstration of the need for some “art” before one can apply the “science” of forecasting. The chart below shows the past several years of stock price history for AMR Corporation.

For comparison to the result calculated in Section Two of this paper, I want to develop a forecast for the AMR stock price on October 15, 1998 (the day the 6-month option expires). In the remainder of this section, I detail the steps taken to produce a forecast for this particular problem. Note: While the problem at hand is to generate a forecast for AMR’s stock price, the steps detailed here are useful for a multitude of business problems that demonstrate compounded percent increases (i.e., exponential growth). Examples include growth trends in demand, sales, customers, distribution, etc.

**Step #1: Plot the historical data.** This sounds simple enough, but is easy to ignore. According to David Dickey, statistics guru from N.C. State, to generate a forecast without a plot is to commit “statistical malpractice”. Statistically sound forecasting techniques can produce illogical results simply because the parameters are not appropriate for a given series of data. For example, a standard exponential smoothing model, with alpha=0.10 (approximately the default alpha in SAS™), applied to the AMR stock price data yields a 6 month forecast of $120 (a 19% drop!). The reason for this is that the most recent 7 points are below the peak; with alpha=0.10, this model “assumes” that this downward trend will continue. A plot of the data will show this, and the user will see that alpha=0.10 is far too high for this data.

**Step #2: Screen for Outliers.** The next step of this process is to screen for outliers. I do NOT mean outliers in the usual statistical sense (high variance from the mean, or disproportionately large impact on the coefficients of the model). The data points to screen for are those that, for whatever reason, have little or no predictive value. Such points need to be identified by someone who knows the data (not simply screened by a computer). In the AMR example, the % change in stock price on the day of a merger announcement, or an accident, may have no predictive value. In this data series, however, there were no such points, and no outliers were removed.

**Step #3: If the series is not “stationary”, transform it into one that is.** A stationary data series is one with a constant mean, a constant variance, and a constant autocorrelation function (more on this shortly). How do you determine if your series is stationary? Below are indicators of non-stationarity:
1) A non-constant mean (if your series has a trend, it’s not stationary).
2) A non-constant variance – statisticians refer to this as “heteroskedasticity” (if the “spread” of your series is not constant over time, it’s not stationary).
3) A slowly decaying autocorrelation function (ACF). The ACF is just the correlation of the series with various lags of itself (hence the name “autocorrelation”). A plot of the ACF is a standard output of the ARIMA procedure in SAS.
4) A Dickey-Freeman Unit Root test with a high p-value. Fortunately, this is a pre-coded macro in SAS (%dftest) that returns a p-value corresponding to the level of confidence with which you can say a given series is stationary (similar in this way to a regression p-value).

Now let’s look at the AMR dataset. It turns out that this series fails all four tests above (it shows a trend with a non-constant variance, a slowly decaying ACF, and a Dickey-Freeman p-value of 0.99). Clearly this series needs to be transformed. Generally, a log transformation will take care of non-constant variance, while a first difference (the discrete version of a first derivative) will take care of a trend. Note: SAS offers a wealth of information about autocorrelation. As mentioned earlier, the ACF shows the correlation of the series with various lags of itself. The Partial ACF (PACF) shows the correlation of the series with various lags of itself that cannot be explained by correlations at lower lags (e.g., it shows the correlation between $X_t$ and $X_{t-1}$ that is beyond what can be explained by their mutual correlation to $X_{t-2}$). One other useful piece of information from the ARIMA procedure is the “Q-statistic”. This can be used to perform a hypothesis test for autocorrelation (Null Hypothesis, $H_0$: any observed autocorrelation in the series is due to sampling). In this way, the Q-statistic is similar to the t-statistic in a regression model. After reviewing the information presented in the output of the ARIMA procedure, the user can then select any appropriate terms to add to the forecast model. For example, if the ACF shows a significant “spike” at lag 1, the user should consider adding an autoregressive term of lag 1 to the model. The ARIMA procedure will then calculate a “t-statistic” which indicates the level of confidence with which one can say that the selected term is, in fact, a valid predictor. This process of identifying and selecting terms (autoregressive or moving average) can be repeated multiple times until the user has a model that is appropriate.

Step #4: Check for autocorrelation. Once you have a stationary data series, the next step is to account for autocorrelation. “Accounting for” autocorrelation actually means tweaking your model to eliminate autocorrelation. This can be done by adding lags of the series, or lags of the forecast errors to the model. Lags of the series are called “autoregressive” terms, while lags of the forecast errors are called “moving average” terms. Note: Any repeating patterns (seasonal or otherwise) will show up as autocorrelation. How can you tell if your series has autocorrelation? The ARIMA procedure in SAS offers a wealth of information about autocorrelation. As mentioned earlier, the ACF shows the correlation of the series with various lags of itself. The Partial ACF (PACF) shows the correlation of the series with various lags of itself that cannot be explained by correlations at lower lags (e.g., it shows the correlation between $X_t$ and $X_{t-1}$ that is beyond what can be explained by their mutual correlation to $X_{t-2}$). One other useful piece of information from the ARIMA procedure is the “Q-statistic”. This can be used to perform a hypothesis test for autocorrelation (Null Hypothesis, $H_0$: any observed autocorrelation in the series is due to sampling). In this way, the Q-statistic is similar to the t-statistic in a regression model. After reviewing the information presented in the output of the ARIMA procedure, the user can then select any appropriate terms to add to the forecast model. For example, if the ACF shows a significant “spike” at lag 1, the user should consider adding an autoregressive term of lag 1 to the model. The ARIMA procedure will then calculate a “t-statistic” which indicates the level of confidence with which one can say that the selected term is, in fact, a valid predictor. This process of identifying and selecting terms (autoregressive or moving average) can be repeated multiple times until the user has a model that is appropriate.

Now let’s look at the “transformed” AMR data series (recall that this transformed data series is really measuring the daily % change in the stock price). It turns out, while lags of the series itself have no predictive value, lags of the forecast errors have some marginal predictive value. What does this mean? It means that:

1) the series is not autoregressive in nature. A shock to the system (e.g., a sudden spike up or down in the series) is not followed by a gradual “regression to the mean”.

To transform this series, I took the log of each stock price, then took the first difference of these values (at each point, subtract the previous value). From the original series $X_1$, $X_2$, $X_3$, . . . . , we now have the transformed series $\ln(X_2)-\ln(X_1)$, $\ln(X_3)-\ln(X_2)$, $\ln(X_4)-\ln(X_3)$, . . . . This is the first difference of the “logged” original series. Let’s examine these terms in more detail. $\ln(X_2)-\ln(X_1)$ can be rewritten as $\ln(X_2/X_1)$. If $X_2$ is today’s price and $X_1$ is yesterday’s price, the ratio $(X_2/X_1)$ is equal to one plus the daily % change. For example, if $X_2$ is 1% higher than $X_1$ the ratio $(X_2/X_1)$ is 1.01. $\ln(1.01) = 0.01$ (to 5 decimals). So this transformed series is really measuring the daily % change. It turns out that the average of the transformed series is 0.000428. This means that the average daily change in stock price is 0.000428, or 0.0428%. This daily return, after accounting for non-trading days like weekends, will produce an annual return of 12%.
series jumps up one day, it is no more likely to come back down the next day than it is to jump up again. Given that this transformed series is based on a stock price, one would not expect it to display autoregressive behavior (autoregressive behavior in a stock price would stand the “Random Walk” theory on its head!).

2) the series has a “moving average” characteristic. A shock to the system is simply incorporated into a “new mean”. This series shows high volatility around a very slowly changing mean – a classic “moving average signature”. The slowly changing mean in this “transformed” series is the daily % change in the stock price – this % change has increased very slowly over the last few years.

Our model will be enhanced if we can account for this “moving average” behavior. One way would be to add “moving average terms” (lags of the forecast errors) to the model. Since there would be several such terms, a simpler way is to use an “exponentially weighted moving average” model (this technique is also known as “exponential smoothing”).

The FORECAST procedure in SAS allows selection of an exponential smoothing model, as well as the smoothing parameters. The parameters include: NSTART - the number of observations used for the initial average, and WEIGHT - the “weight” to place on each new observation (known as the “smoothing alpha”). It is critical that the user understands the impact of these two parameters (recall the example from Step #1). One other parameter to be aware of is the TREND. SAS allows the user to select a constant, linear, or quadratic trend. For a stationary series, we want to use the constant trend. Why? Recall that a stationary series displays no linear trend. Forcing an exponential smoothing model with a trend onto a series with no trend will result in the extrapolation of variance (interpreted as a local trend) into a global trend.

The “transformed” data series, measuring the daily % change in AMR’s stock price, has a constant mean. A “constant” mean in the context of an exponential smoothing model actually allows for gradual changes in the mean, as long as there is no discernible trend. What would happen if we tried to fit an exponential smoothing model with a linear trend? The model would forecast a % change that is increasing or decreasing as a function of time! This is clearly unreasonable for this series. Even if such a model provided a good “fit” (as measured by a relatively low standard deviation in the forecast), it would actually be (with the rarest of exceptions) an example of overfitting. One of the most common mistakes in statistical analysis is to interpret movements due to sampling error as movements that have predictive value (i.e., to interpret random movements as systemic movements). Fitting a trend to this series would amount to making just such an interpretation.

REVIEW AND RESULTS:

How good is this new technique – Option Pricing with a Time Series? Let’s apply it to the AMR data series. First we follow the 4 steps just discussed to produce a forecast (recall that these steps are very applicable to many compound growth problems).
1) Plot the data – this gives an indication about linearity, variance, and appropriate “weightings”.
2) Screen for outliers – all points in this series were deemed to have predictive value (no outliers).
3) Transform the series using a log, and then a first difference – creating a new (stationary) series that measures daily % change in stock price.
4) Check for autocorrelation – as expected, this series has “moving average” behavior, and no “autoregressive” behavior. We can now use an exponential smoothing model. The actual parameters chosen for this problem (refer to section III for details) were NSTART=100, WEIGHT=0.005, and TREND=constant.

The model produces a forecast for October 15, 1998 (the date the 6 month option expires) of $167.77, with a standard deviation of $12.48. Applying steps two and three from the binomial model described in section II, we can calculate an option value of $18.15! This is a bit higher than the “market” price of $16.00. Why? For this forecast, I used the most recent three years of data. Since the stock has increased quite a bit in the past three years, this is reflected in the forecast. This is almost certainly an
overstatement of what the stock is expected to do in the next 6 months.

Post note: Between the date of my data collection (April 24, 1998) and the date this paper was submitted (July 10, 1998), AMR’s stock price split two-for-one. This, of course, has no relevance to the option price as of April 24, which is the focus of my paper (also, the terms of an option are adjusted for stock splits). However, for those of you who are stock watchers, my forecast for AMR’s stock price in this paper of $167.77 does not reflect the split. Barring any further splits between July 10 and October 15, my time series analysis says: “look for a price of approx. $84”.

APPENDIX I - OVERVIEW OF FORECASTING.

Forecasting is a critical aspect of virtually every business. Quantitative forecasting can be divided into two categories: causal and time series. Causal forecasting involves the use of “explanatory variables” in the forecast. Time series forecasts project trends from the past into the future. For example, if we want to forecast the demand at a particular restaurant, we could use causal or time series models. A time series model would use historical data for that same restaurant (with trends, seasonality, other fluctuations, some weighting & screening scheme for the data points, etc.). A causal model could also incorporate estimates of economic / demographic changes and competitive behavior. At first glance, one might think that causal models are always better since they address factors that a time series model ignores. However, these explanatory variables may be subject to their own forecast error, and the interactions between variables can be quite difficult to measure and understand. The technique described in this paper is an application of a time series model.

APPENDIX II - OVERVIEW OF STOCK OPTIONS - CONTINUED.

An “American” option allows the holder to exercise the option at anytime before (or on) the expiration date, while a “European” option may be exercised only on the expiration date. Because of this, an American option tends to have a higher value than a European option with otherwise identical attributes.

Note that I did not specify “American” or “European” options for my focus. It turns out that it is never advantageous to exercise a call option on a non-dividend paying stock early. This makes sense because a holder should always choose to hold this option until the last possible moment (or sell it). Because of this, an American call option on a non-dividend paying stock has the same value as its European counterpart (i.e., the “option” to exercise this option early has no value).

Stock options are gaining popularity in Corporate America for a variety of reasons. One of the major reasons is that shareholders are increasingly demanding that executive compensation be commensurate with (or at least related to) company performance. Stock options have some other attractive features for a company too. Benefits include – employee morale, reduced turnover, and accounting / tax advantages.

In addition to increased interest in actual stock options, Corporate America is also increasingly using “option pricing” techniques to analyze their business. For example, a railroad company has some freight cars that are not currently in use. Should it sell these cars, or let them sit in the rail yard unused? The “option” here is the option to start using this railcar at a purchase price of zero. The cost of this option is the cost of maintenance & storage, and the opportunity cost of not selling the car to generate some revenue. If business picks up, this “option” will certainly have some value; if the railcar is never used, this option will provide zero value.

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